Firms

ECON 105: Intro to Mathematical Economics

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Old midterm question

1. Maria's hicksian demand for coffee is given by $h_c(p_c, p_t, \bar{U}) = \sqrt[5]{\bar{U}^6 \left(\frac{2p_t}{3p_c}\right)^3}$ where p_c is the price of coffee and p_t is the price of tea. The compensated own price elasticity ε^c_{c,p_c} and the compensated cross price elasticity ε^c_{c,p_t} are ?

   (a) $\varepsilon^c_{c,p_c} = 3/5$ and $\varepsilon^c_{c,p_t} = 3/5$

   (b) $\varepsilon^c_{c,p_c} = -3/5$ and $\varepsilon^c_{c,p_t} = 3/5$

   (c) $\varepsilon^c_{c,p_c} = -1$ and $\varepsilon^c_{c,p_t} = 1$

   (d) $\varepsilon^c_{c,p_c} = 5/6$ and $\varepsilon^c_{c,p_t} = 6/5$

   (e) $\varepsilon^c_{c,p_c} = -1/3$ and $\varepsilon^c_{c,p_t} = 1/3$

Old midterm question

9. Suppose $E(p_x, p_y, \underline{U}) = 2\underline{U}^{\frac{1}{2}} p_x^{\frac{1}{2}} p_y^{\frac{1}{2}}$ the Marshallian demand of good y is:

(a) $y(p_x, p_y, I) = \frac{I}{2p_y}$

(b) $y(p_x, p_y, I) = \frac{I}{2p_y^{\frac{1}{2}}}$

(c) $y(p_x, p_y, I) = \frac{I^2}{4p_y^2}$

(d) $y(p_x, p_y, I) = \underline{U}^{\frac{1}{2}} p_x^{-\frac{1}{2}} p_y^{\frac{1}{2}}$

Firm problem vs. consumer problem

- Consumers: how much of each good to purchase
- Firms: multiple stages
  - Choose inputs for production based on costs (cost minimization)
  - Choose how much to produce (profit maximization)

$$\text{Production function} \longrightarrow \text{Cost function} \longrightarrow \text{Profit function}$$

Production function

- **production function**: amount of output a firm can produce given inputs capital (k) and labor (l) is determined by a

$$q = f(k, l)$$

- E.g., a baker making 100 loaves of bread per day must decide how many workers to hire and how many ovens to buy
- Firms choose combination of k and l given their costs
  - The cost of labor is the **wage** rate paid to employees w
  - The cost of capital if the **rental** rate of the machines v

Cost minimization problem

Choose inputs k, l to produce a given quantity q at the lowest cost:

$$\min_{k,l \geq 0} \quad vk + wl$$
$$\text{s.t. } f(k,l) = q$$

- Solution is the pair k^c, l^c, called **contingent demand** (depends on quantity q the firm needs to produce)

$$k^c(w, v, q), \ l^c(w, v, q)$$

- Constrained optimization $\longrightarrow$ Lagrangian $\longrightarrow$ optimality condition

Cost minimization set-up

$$\min_{k,l \geq 0} \quad vk + wl$$
$$\text{s.t. } f(k,l) = q$$

$$\mathcal{L} = vk + wl + \lambda \left(q - f(k,l) \right)$$

FOC:

$$\frac{\partial \mathcal{L}}{\partial l} : w - \frac{\partial f}{\partial l} = 0 \qquad \Longrightarrow w = \frac{\partial f}{\partial l} \tag{1}$$
$$\frac{\partial \mathcal{L}}{\partial k} : v - \frac{\partial f}{\partial k} = 0 \qquad \Longrightarrow v = \frac{\partial f}{\partial k} \tag{2}$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} : q - f(k,l) = 0 \qquad \Longrightarrow f(k,l) = q \tag{3}$$

Famliar optimality condition

$$RTS_{l,k} = \frac{\frac{\partial f}{\partial l}}{\frac{\partial f}{\partial k}} = \frac{w}{v}$$

- **Marginal Rate of Technical Substitution** $(RTS_{l,k})$: relative productivities of labor and capital
  - Tells us the trade off capital and labor, keeping production unchanged
- **Input price ratio**: ratio of wage rate to rental rate $\frac{w}{v}$
  - Relative cost of labor compared to capital

Solve for l or k and plug into production constraint $f(k,l) = q$

Marginal product of inputs

- Amount of additional production from increasing an input (labor or capital)
  - Marginal product of labor (MPL)$= \frac{\partial f(k,l)}{\partial l}$
  - Marginal product of capital (MPK)$= \frac{\partial f(k,l)}{\partial k}$

$$RTS = \frac{MPL}{MPK}$$

Returns to scale

If a firm increases all inputs by the same amount t, how much does output increase?

- $f(t*k, t*l) = t*f(k,l) \implies$ constant returns to scale
- $f(t*k, t*l) > t*f(k,l) \implies$ increasing returns to scale
- $f(t*k, t*l) < t*f(k,l) \implies$ decreasing returns to scale

Note that constant returns to scale means the production function is homogeneous of degree 1

Elasticity of substitution

- How substitutable are labor and capital?
  - E.g., human autoworkers vs. robots
- Compute % change in $\frac{k}{l}$ relative to change in RTS

$$\sigma = \frac{\partial(\frac{k}{l})}{\partial RTS} * \frac{RTS}{(\frac{k}{l})}$$

$$\sigma = \frac{d\ln(k/l)}{d\ln(RTS)}$$

Old exam question

8. Consider the following production function $Q = K^{\frac{2}{3}} L^{\frac{1}{2}}$, where K represents the units of capital employed at the production facility, L is the number of labor hours employed and Q is the total production. What is the elasticity of substitution ?

Old exam question

Emma's bike company has the following production function: $q = \alpha\left(aK^{\frac{1}{2}} + bL^{\frac{1}{2}}\right)$, with $a > 0$ and $b > 0$. Answer the following questions.

3. What is Emma's Marginal Rate of Technical Substitution (How many units of K is she willing to sacrifice to get one more unit of labor, holding production constant? Put labor on the horizontal axis and capital on the vertical axis.). (4 Points)