Income & Substitution Effects, Continued

ECON 105: Intro to Mathematical Economics

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Old midterm question

1. Suppose the Marshallian demands for goods X and Y are given as $x^* = \frac{5I}{5p_x+3p_y}$ and $y^* = \frac{3I}{5p_x+3p_y}$. And the indirect utility function is given as $V(p_x, p_y, I) = \frac{15I}{5p_x+3p_y}$. Which of the following is the Hicksian demands for goods X and Y?

   (a) $h_x = \frac{\bar{U}}{3}$, $h_y = \frac{\bar{U}}{5}$

   (b) $h_x = \frac{\bar{U}}{5}$, $h_y = \frac{\bar{U}}{3}$

   (c) $h_x = \frac{I}{p_x}$, $h_y = 0$

   (d) $h_x = \frac{3p_y}{p_x}$, $h_y = \frac{5p_x}{p_y}$

   (e) $h_x = \frac{5p_y}{p_x}$, $h_y = \frac{3p_x}{p_y}$

Connecting Marshallian and Hicksian demand

Duality of Marshallian (g) & Hicksian (h) Demand

$$\bar{U} = V(p_x, p_y, I)$$

$$I = E(p_x, p_y, \bar{U})$$

$$g(p_x, p_y, E(\cdot)) = h(p_x, p_y, \bar{U})$$

$$h(p_x, p_y, V(\cdot)) = g(p_x, p_y, I)$$

$$h_x(\cdot) = \frac{\partial E(\cdot)}{\partial p_x}$$

$$g_x(\cdot) = -\frac{\frac{\partial V(\cdot)}{\partial p_x}}{\frac{\partial V(\cdot)}{\partial I}}$$

Hicksian: Cobb–Douglas

$$u(x_1, x_2) = x_1^a x_2^b, \quad a, b > 0$$

Expenditure function:

$$E(p_1, p_2, \bar{U}) = (a+b)\, \bar{U}^{\frac{1}{a+b}}\, a^{-\frac{a}{a+b}}\, b^{-\frac{b}{a+b}}\, p_1^{\frac{a}{a+b}}\, p_2^{\frac{b}{a+b}}$$

Hicksian demands:

$$h_1 = \frac{a}{a+b}\, \frac{E(p, \bar{U})}{p_1}, \qquad h_2 = \frac{b}{a+b}\, \frac{E(p, \bar{U})}{p_2}$$

Marshallian demands:

$$g_1 = \frac{a}{a+b} \cdot \frac{I}{p_1}, \quad g_2 = \frac{b}{a+b} \cdot \frac{I}{p_2}$$

Hicksian: Perfect Substitutes

$$u(x_1, x_2) = ax_1 + bx_2, \quad a, b > 0$$

Hicksian demands:

$$(h_1, h_2) = \begin{cases} \left(\dfrac{\bar{U}}{a}, 0\right) & \text{if } \dfrac{p_1}{a} < \dfrac{p_2}{b} \\ \left(0, \dfrac{\bar{U}}{b}\right) & \text{if } \dfrac{p_1}{a} > \dfrac{p_2}{b} \\ \text{any bundle with } ax_1 + bx_2 = \bar{U} & \text{if } \dfrac{p_1}{a} = \dfrac{p_2}{b} \end{cases}$$

Marshallian demands:

$$(g_1, g_2) = \begin{cases} \left(\dfrac{I}{p_1}, 0\right) & \text{if } \dfrac{a}{b} > \dfrac{p_1}{p_2} \\ \left(0, \dfrac{I}{p_2}\right) & \text{if } \dfrac{a}{b} < \dfrac{p_1}{p_2} \end{cases}$$

Hicksian: Perfect Complements

$$u(x_1, x_2) = \min\{ax_1, bx_2\}\,, \quad a, b > 0$$

Hicksian demands:

$$h_1 = \frac{\bar{U}}{a}, \qquad h_2 = \frac{\bar{U}}{b}$$

Expenditure:

$$E(p, \bar{U}) = \left(\frac{p_1}{a} + \frac{p_2}{b}\right)\bar{U}$$

Marshallian demands:

$$g_1 = \frac{bI}{bp_1 + ap_2}, \quad g_2 = \frac{aI}{bp_1 + ap_2}$$

Hicksian: Quasi–Linear

$$u(x_1, x_2) = f(x_1) + bx_2$$

$$p_1 x_1 + p_2 \left(\frac{\bar{U} - f(x_1)}{b} \right) \quad \Rightarrow \quad \frac{f'(x_1)}{b} = \frac{p_1}{p_2}$$

Hicksian demands:

$$h_1 = x_1 \text{ s.t. } \frac{f'(x_1)}{b} = \frac{p_1}{p_2}, \qquad h_2 = \frac{\bar{U} - f(x_1)}{b}$$

Marshallian demands:

$$(g_1, g_2) = \begin{cases} g_1 \quad s.t. \quad \dfrac{f'(x_1)}{b} = \dfrac{p_1}{p_2}, \\ g_2 = \dfrac{I - p_1 g_1}{p_2} \end{cases}$$

Slutsky equation

The **Slutsky equation** splits the overall impact of a price change on demand into income and substitution effects:

$$\underbrace{\frac{\partial x_m(p_x, p_y, I)}{\partial p_x}}_{\text{Total Effect}} = \underbrace{\frac{\partial x_h(p_x, p_y, \underline{\mathsf{U}})}{\partial p_x}}_{\text{Sub. Effect}} \underbrace{- \frac{\partial x_m(p_x, p_y, I)}{\partial I} x_m}_{\text{Income Effect}}$$

Large price changes

For “large” price changes, don’t use Slutsky:

$$Total_x = \frac{g_x^{new} - g_x^{old}}{p_x^{new} - p_x^{old}}$$

$$Total_y = \frac{g_y^{new} - g_y^{old}}{p_x^{new} - p_x^{old}}$$

$$Substitution_x = \frac{h_x^{new} - h_x^{old}}{p_x^{new} - p_x^{old}}$$

$$Substitution_y = \frac{h_y^{new} - h_y^{old}}{p_x^{new} - p_x^{old}}$$

$$Income_x = Total_x - Substitution_x$$

$$Income_y = Total_y - Substitution_y$$

Small price changes

Use Slutsky for "marginal" price changes

"Own-price":

$$\underbrace{\frac{\partial g_x}{\partial p_x}}_{\text{Total Effect}} = \underbrace{\frac{\partial h_x}{\partial p_x}}_{\text{Sub. Effect}} \underbrace{-\frac{\partial g_x}{\partial I} g_x}_{\text{Income Effect}}$$

"Cross-price":

$$\frac{\partial g_y}{\partial p_x} = \frac{\partial h_y}{\partial p_x} - \frac{\partial g_y}{\partial I} g_x$$

Elasticities

- Economists often use **elasticities** to quantify price effects
  - Focus on proportional effect of a change in one variable on another
  - **Unit-free** $\implies$ compare responses across consumers

- Definition: percentage change in y when x changes 1%

$$e_{y,x} = \frac{\frac{\Delta y}{y}}{\frac{\Delta x}{x}} = \frac{\Delta y}{\Delta x} \cdot \frac{x}{y}$$

$$\implies e_{y,x} = \frac{\partial y}{\partial x} \cdot \frac{x}{y}$$

$$e_{y,x} = \frac{d\ln(y)}{d\ln(x)}$$

Elasticity Examples

Find the elasticity of y with respect to x for the following functions

- $y = a + bx$

- $y = ax^b$

- $\ln(y) = \ln(a) + b\ln(x)$

Demand elasticities

- Price elasticity: a 1% increase in p_x changes x by e%

$$e_{x,p_x} = \frac{\partial x}{\partial p_x} \cdot \frac{p_x}{x}$$

- Cross-price elasticity: a 1% increase in p_y changes x by e%

$$e_{x,p_y} = \frac{\partial x}{\partial p_y} \cdot \frac{p_y}{x}$$

- Income elasticity: a 1% increase in I changes x by e%

$$e_{x,I} = \frac{\partial x}{\partial I} \cdot \frac{I}{x}$$

Cobb-Douglas Elasticity

- Consider a consumer with a utility function $u(x, y) = xy$
- This consumer faces prices p_x, p_y, and has income I.

1. Find the marshallian demands
2. Find the hicksian demands
3. Find price and income elasticities for both marshallian and hicksian demands

Cobb-Douglas Elasticity

Practice: perfect substitutes

Suppose $u(x, y) = 4x + 7y$ and prices are $p_x = 3, p_y = 8$ and $I = 24$. The price of x then goes up to 4. What are the total, income, and substitution effects for x, y?

Practice: perfect substitutes

Practice: perfect complements

Suppose $u(x, y) = \min\{2x, 3y\}$ and prices are $p_x = 4, p_y = 6$ and $I = 72$. The price of good x then falls to 2. What are the total, income, and substitution effects?

Practice: perfect complements